MATH2060B Tutorial 4

Topic: Taylor's Theorem

Q4 Show that if $x>0$, then $1+\frac{1}{2}x-\frac{1}{8}x^2 \le \sqrt{1+x} \le 1+\frac{1}{2}x$

Sol Use Taylor's thm 6.4.1 with $n=1,2$ & $x_0=0$.

$f(x)=\sqrt{1+x}$ $\quad f'(x)=\frac{1}{2}(1+x)^{-1/2}$ $\quad f''(x)=-\frac{1}{4}(1+x)^{-3/2}$ $\quad f'''(x)=\frac{3}{8}(1+x)^{-5/2}$

$f(0)=1$ $\quad f'(0)=\frac{1}{2}$ $\quad f''(0)=-\frac{1}{4}$

By Taylor Thm, for $x>0$,

$$① \quad f(x)=\underbrace{f(0)+f'(0)x}_{P_1}+\underbrace{\frac{f''(c_1)}{2}x^2}_{R_1}$$

$$=1+\frac{x}{2}+\frac{f''(c_1)}{2}x^2 \qquad 0<c_1<x$$

$$② \quad f(x)=\underbrace{f(0)+f'(0)x+\frac{f''(0)}{2}x^2}_{P_2}+\underbrace{\frac{f'''(c_2)}{3!}x^3}_{R_2}$$

$$=1+\frac{x}{2}-\frac{1}{8}x^2+\frac{f'''(c_2)}{6}x^3 \qquad 0<c_2<x$$

By above computation, $f''(c_1)<0 \Rightarrow \frac{f''(c_1)}{2}x^2<0$

$① \Rightarrow f(x)\le 1+\frac{x}{2}$

Similarly, $f'''(c_2)>0 \Rightarrow \frac{f'''(c_2)}{6}x^3>0$ for $x>0$

$② \Rightarrow f(x)\ge 1+\frac{x}{2}-\frac{1}{8}x^2$

$\rightsquigarrow$ Together these show that

$$1+\frac{x}{2}-\frac{1}{8}x^2 \le f(x)=\sqrt{1+x} \le 1+\frac{x}{2}$$

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Q9 If $g(x) := \sin x$, show that the remainder term in Taylor's Theorem converges to zero as $n \to \infty$ for each fixed x_0 and x.

Sol According to Taylor's Thm, remainder term for $g(x) = \sin x$, with fixed $x_0 \neq x$ ($R_n(x_0) = 0 \ \forall n$)

$$R_n(x) = \frac{g^{(n+1)}(c_n)}{(n+1)!}(x - x_0)^{n+1}$$

for some $x_0 < c_n < x$.

Since $g^{(n+1)}(x) = \sin x, \cos x, -\sin x$ or $-\cos x$. we always have $|g^{(n+1)}(c_n)| \leq 1$

$$\Rightarrow |R_n(x)| \leq \frac{|x - x_0|^{n+1}}{(n+1)!}$$

With x and x_0 fixed, consider the sequence $a_n \in \mathbb{R}$, defined by $a_n = \dfrac{|x - x_0|^{n+1}}{(n+1)!}$. Since (a_n) satisfies

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|x - x_0|^{n+2}}{(n+2)!} \cdot \frac{(n+1)!}{|x - x_0|^{n+1}}$$

$$= \lim_{n\to\infty} \frac{|x - x_0|}{n+2}$$

$$= 0 < 1$$

Ratio test (Thm 3.2.11) $\Rightarrow \lim_{n\to\infty} a_n = 0$

$\rightsquigarrow |R_n(x)| \leq a_n \Rightarrow R_n(x) \xrightarrow[\text{as } n \to \infty]{} 0$ by squeeze Thm for any fixed x_0 & x. //

Q10 Let $h(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Show that $h^{(n)}(0) = 0 \ \forall n \in \mathbb{N}$.

Conclude that the remainder term in Taylor's Thm for $x_0 = 0$ does not converge to zero as $n \to \infty$ for $x \neq 0$.

Sol. Observe that $h(x)$ is continuous at $x=0$, $\because \frac{1}{x^2} \to \infty$ as $x \to 0$.

Claim 1. $\lim_{x \to 0} \frac{h(x)}{x^k} = 0$ for all $k \in \mathbb{N}$

Claim 2. For $x \neq 0$, $h^{(n)}(x) = p_n(x) \frac{e^{-1/x^2}}{x^{3n}}$ for some polynomial p_n.

$$= p_n(x) \frac{h(x)}{x^{3n}}$$

To prove $h^{(n)}(0) = 0$ (for all n) from the two claims: by induction

For $n=1$ $\quad \lim_{x \to 0} \frac{h(x) - h(0)}{x} = \lim_{x \to 0} \frac{h(x)}{x} = 0$ by Claim 1.

$\Rightarrow h'(0)$ exists and equals 0.

Suppose $h^{(n)}(0) = 0$, then

$$\lim_{x \to 0} \frac{h^{(n)}(x) - h^{(n)}(0)}{x} = \lim_{x \to 0} \frac{h^{(n)}(x)}{x} \quad \text{(induction hypothesis)}$$

$$= \lim_{x \to 0} p_n(x) \frac{h(x)}{x^{3n+1}} \quad \text{(claim 2)}$$

$$= \left(\lim_{x \to 0} p_n(x)\right) \cdot \left(\lim_{x \to 0} \frac{h(x)}{x^{3n+1}}\right)$$

$$= 0 \quad \text{(claim 1)}$$

$\Rightarrow h^{(n+1)}(0)$ exists and equals 0

Result then follows by induction.

Because $h^{(k)}(0) = 0 \ \forall k \Rightarrow$ Taylor polynomial P_n for $h(x)$ at $x=0$ is always the zero function. $\forall n \in \mathbb{N}$

In particular, for (fixed) $x \neq 0$, the remainder term $R_n(x) = h(x)$ is a non-zero constant for all n and so does not converge to zero as $n \to \infty$.

Claim 1. $\lim_{x\to 0} \frac{h(x)}{x^k} = 0$ for all $k \in \mathbb{N}$

Pf By induction over k.

When $k=1$, $\lim_{x\to 0} \frac{h(x)}{x} = \lim_{x\to 0} \frac{x^{-1}}{e^{x^{-2}}}$ ("$\frac{\infty}{\infty}$")

$$= \lim_{x\to 0} \frac{-x^{-2}}{-2x^{-3}e^{x^{-2}}} \quad \text{(L'hopital)}$$

$$= \frac{1}{2}\lim_{x\to 0} x\, h(x)$$

$$= 0.$$

Suppose statement holds for k, then

$$\lim_{x\to 0} \frac{h(x)}{x^{k+1}} = \lim_{x\to 0} \frac{x^{-(k+1)}}{e^{x^{-2}}} \quad \left(\text{"}\frac{\infty}{\infty}\text{"}\right)$$

$$= \lim_{x\to 0} \frac{-(k+1)\,x^{-(k+2)}}{-2x^{-3}e^{x^{-2}}} \quad \text{(L'hopital)}$$

$$= \frac{k+1}{2}\lim_{x\to 0} \frac{h(x)}{x^{k-1}}$$

$$= \frac{k+1}{2}\lim_{x\to 0} x \cdot \frac{h(x)}{x^k} \quad \text{(induction hyp.)}$$

$$= 0$$

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Claim 2 For $x \neq 0$, $h^{(n)}(x) = p_n(x)\frac{e^{-1/x^2}}{x^{3n}}$ for some polynomial p_n.

$$= p_n(x)\frac{h(x)}{x^{3n}}$$

Pf (again by induction)

For $n=1$, $h'(x) = \left(e^{-x^{-2}}\right)' = -(-2)x^{-3}e^{-x^{-2}}$

$$= \frac{2h(x)}{x^3}$$

i.e. $h'(x)$ is of the form $p_1(x)\frac{h(x)}{x^3}$.

Suppose statement holds for $n \in \mathbb{N}$, then

$$h^{(n+1)}(x) = \left(p_n(x)\frac{h(x)}{x^{3n}}\right)' = p_n'(x)\frac{h(x)}{x^{3n}} - 3np_n(x)\frac{h(x)}{x^{3n+1}} + p_n(x)\frac{h'(x)}{x^{3n}}$$

$$= p_n'(x)\frac{h(x)}{x^{3n}} - 3np_n(x)\frac{h(x)}{x^{3n+1}} + 2p_n(x)\frac{h(x)}{x^{3(n+1)}}$$

$$= \left[x^3p_n'(x) - 3nx^2p_n(x) + 2p_n(x)\right]\frac{h(x)}{x^{3(n+1)}}$$

which is of the form $p_{n+1}(x)\frac{h(x)}{x^{3(n+1)}}$. Claim 2 follows by induction.

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Q22 The equation $\ln x = x-2$ has two solutions. Approximate them using Newton's method. What happens if $x_1 := \frac{1}{2}$ is the initial point?

Sol Define $f(x) = \ln x - x + 2$ on $x > 0$.

→ Observe that $f(x) \to -\infty$ as $x \to 0^+$

$f(1) = 1$

$f(e^2) = 4 - e^2 < 0$.

$\text{IVT} \overset{f \text{ cts}}{\Longrightarrow}$ f has a root in $(0,1)$ & another in $(1,8)$

→ From $f'(x) = \frac{1}{x} - 1$, we see that

① f has a (relative) max at $x=1$ (first derivative test)

② These are the only zeros of f. (Rolle's Thm)

→ To apply Newton method, make initial guesses x_1 and define iteratively $x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}$

e.g. $x_1 = 2$

$x_2 = 2 - \frac{f(2)}{f'(2)} \approx 3.38629436112$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 3.1499383938$

$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx 3.14619425703$

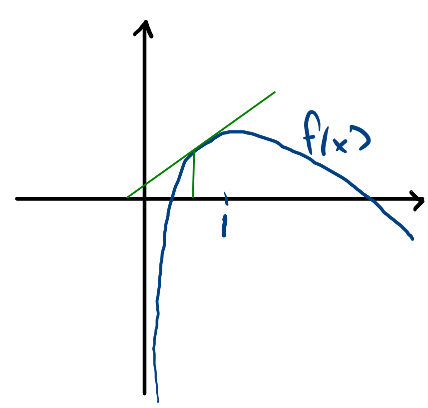
$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} \approx 3.14619322062$

(up to 11 decimal places)

e.g. $x_1 = \frac{1}{2} \Rightarrow x_2 \approx -0.307$,

and x_3 undefined because

f only defined on $x > 0$



e.g. $x_1 = 0.3$

$\Rightarrow x_6 \approx 0.1585943391 4$ (correct up to 9 decimal places)